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Topologic Proofs of Some Combinatorial Theorems

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During the last 50 years several combinatorial theorems have been proved which have provided elegant proofs of a number of fundamental results in topology. These include Sperner's Lemma, Tucker's Lemma, and Kuhn's Cubical Sperner Lemma, which have been applied to the Brouwer Fixed Point Theorem, as well as a number of results involving antipodal properties of continuous mappings.

In this paper the reverse process is used to find topologic proofs of several combinatorial results. In response to several questions raised by Kuhn, a proof of Sperner's Lemma from the Brouwer Fixed Point Theorem is given, as is a proof of Tucker's Lemma from a topologic non-existence theorem for certain continuous mappings of an n-ball to its boundary. In the final section, a new labeling theorem for the n-cube, which is equivalent to Tucker's Lemma, is presented, and is proved by using topologic methods.

1. INTRODUCTION

In 1928, Sperner [6] presented a purely combinatorial lemma concerned with the nature of certain labelings of the vertices of subdivisions of simplexes. The following year Knaster, Kuratowski and Mazurkiewicz [3] employed this lemma to prove the Brouwer Fixed Point Theorem. Since that time several other combinatorial results [2, 4, 8] have been proved which have also been shown to imply the Brouwer Fixed Point Theorem as well as a number of other topologic results.

In particular, in 1945, Tucker [8] proved a combinatorial labeling lemma for subdivisions of the n-cube which involved the antipodal structure of the cube. He used this result to prove several theorems concerning continuous mappings of topologic objects in which some antipodal structure was considered.

Another of the combinatorial results mentioned above is a theorem due to Kuhn [4] which is an analog of Sperner's Lemma for n-cubes, as opposed to n-simplexes. Kuhn demonstrates that his Strong Cubical

Sperner Lemma not only implies the Brouwer Theorem for n-cubes, but is equivalent to it. As he points out, this raises a number of questions regarding the equivalence of various combinatorial results and corresponding topologic results.

The purpose of this paper is to answer in the affirmative two of the questions posed by Kuhn; namely,

(i) Can Sperner's Lemma be derived from the Brouwer Fixed Point Theorem, and

(ii) Is it possible to derive Tucker's Lemma from an Antipodal Point Theorem?

We also present a new labeling result for the n-cube which is equivalent to Tucker's Lemma, and which can be derived by topologic considerations.

2. Sperner's Lemma

We begin by stating both of the results whose equivalence will be proved:

Brouwer Fixed Point Theorem for Simplexes. *Every continuous mapping of an n-simplex into (all or part of) itself has at least one fixed point.*

To state Sperner's Lemma we will need a bit of terminology. Let σ^n be an n-simplex, and let $\mathscr{S} = \{\sigma_1{}^n, \sigma_2{}^n, \ldots, \sigma_t{}^n\}$ be a finite set of n-simplexes. We will call $\mathscr{S}$ a subdivision of σ^n if the following hold:

(i) $$\bigcup_i \sigma_i{}^n = \sigma^n, \tag{1}$$

(ii) $$\text{for } i \neq j,\ \sigma_i{}^n \cap \operatorname{int}(\sigma_j{}^n) = \varnothing, \tag{2}$$

(iii) if p is a vertex of $\sigma_i{}^n$, for some i, then for all j, if $p \in \sigma_j{}^n$, p is a vertex of $\sigma_j{}^n$. (3)

We will use barycentric coordinates to denote the points in a simplex. Then, an n-simplex is given by:

$$\sigma^n = \left\{X = (x_0, x_1, \ldots, x_n) \mid x_i \geqslant 0, 0 \leqslant i \leqslant n \text{ and } \sum_{i=0}^{n} x_i = 1\right\}.$$

Let $\mathscr{S}$ be a subdivision of $\sigma_i{}^n$ and V the set of vertices of the subdivision.

By a *proper labeling* of V, we will mean a function L,

$$L: V \to I_n, \tag{4}$$

where I_n is the set of integers $\{0, 1,..., n\}$, subject to the condition that, if $L(v) = i$, where $v = (x_0, x_1, ..., x_n)$, then $x_i > 0$.

Notice that from this condition each of the $n + 1$ vertices of σ^n carries a distinct label. Also, any point on a boundary face must carry one of the labels of the vertices in the carrier of that face.

With these definitions we can now state:

SPERNER'S LEMMA [6]. *For any subdivision and any proper labeling of a simplex, there is at least one simplex in the subdivision whose vertices carry a complete set of labels.*

In order to prove Sperner's Lemma from the Brouwer Fixed Point Theorem, we will employ the following scheme: for any subdivision and any proper labeling of it, a function will be constructed. This function will be shown to map the simplex into itself and to be continuous, thus satisfying the hypotheses of the Brouwer Theorem. It will finally be shown that the only points which remain fixed under this mapping are the barycenters of simplexes in the subdivision which carry a complete set of labels.

Let σ^n be an n-simplex and $\mathscr{S}$ a subdivision of it. Let

$$V = \{v_1, v_2, ..., v_r\}$$

be the set of all the vertices of the elements of $\mathscr{S}$, where each v_i has barycentric coordinates

$$v_i = (x_{i0}, x_{i1}, ..., x_{in}).$$

Let $L: V \to I_n$ be a proper labeling. For the vertices in V, we will denote $L(v_i)$ by $l(i)$.

We will define the function described above first on V, and then extend it to all of σ^n. For this, let $f: V \to \sigma^n$ be defined by:

$$f(v_i) = (x'_{i0}, x'_{i1}, ..., x'_{in}) = v_i',$$

$$x'_{ij} = \begin{cases} x_{ij} - \epsilon, & \text{for } j = l(i), \\ x_{ij} + \dfrac{\epsilon}{n}, & \text{for } j \neq l(i), \end{cases} \tag{5}$$

where

$$\epsilon = \min_{1 \leqslant i \leqslant r} \{x_{il(i)}\}.$$

Clearly,

$$f(v_i) \in \sigma^n \qquad \text{for all} \quad v_i \in V.$$

We extend f to all of σ^n by extending it to each of the simplexes in the subdivision.

Let $\sigma_i^n \in \mathscr{S}$, with vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_n}$. Then

$$\sigma_i^n = \{X \in \sigma^n \mid X = \lambda_0 v_{i_0} + \lambda_1 v_{i_1} + \cdots + \lambda_n v_{i_n}\},$$

where

$$\lambda_j \geqslant 0 \qquad \text{for} \quad 0 \leqslant j \leqslant n,$$

and

$$\sum_{j=0}^{n} \lambda_j = 1.$$

Then, if $X = \sum_{j=0}^{n} \lambda_j v_{i_j}$, let

$$f(X) = \sum_{j=0}^{n} \lambda_j f(v_{i_j}). \tag{6}$$

It is easily seen that this is well defined and continuous for all of the points in σ^n. We need only observe that, if $X \in \sigma_i^n \cap \sigma_j^n$, $i \neq j$, then the value of f computed for X viewed as a point in σ_i^n is equal to the value of f computed for X viewed as a point in σ_j^n.

The function f satisfies the hypotheses of the Brouwer Fixed Point Theorem. Thus, there must be at least one point $X \in \sigma^n$ such that $f(X) = X$. We can characterize a fixed point of f by the following:

LEMMA 1. *Given an n-simplex σ^n, with subdivision $\mathscr{S}$ and proper labeling L, let f be defined on σ^n by* (5) *and* (6). *If X is a fixed point of f, then X is contained in an element σ_i^n of $\mathscr{S}$ which has a complete set of labels on its vertices.*

Proof. We will actually prove the equivalent statement: if $\sigma_i^n \in \mathscr{S}$ does not have a complete set of labels, then f cannot have a fixed point in σ_i^n. For this, let $\sigma_i^n \in \mathscr{S}$ with vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_n}$, where

$$v_{i_j} = (x_{j_0}, x_{j_1}, \ldots, x_{j_n}).$$

Assume that no vertex of σ_i^n carries the label k, where k is some integer in I_n. From (5), if

$$f(v_{i_j}) = (x'_{j_0}, x'_{j_1}, \ldots, x'_{j_n}),$$

then component $x'_{jk} = x_{jk} + \epsilon/n$, for all $j = 0, 1, 2, \ldots, n$.

Let $X \in \sigma_i^n$. Then

$$X = \sum_{j=0}^{n} \lambda_j v_{i_j},$$

where

$$\lambda_j \geqslant 0, \qquad \text{for all} \quad 0 \leqslant j \leqslant n,$$

and

$$\sum_{j=0}^{n} \lambda_j = 1.$$

Computing $f(X)$ from (6),

$$f(X) = \sum_{j=0}^{n} \lambda_j f(v_{i_j}).$$

The k-th component of $f(X)$ is

$$\sum_{j=0}^{n} \lambda_j(x'_{jk}) = \sum_{j=0}^{n} \lambda_j x_{jk} + \frac{\epsilon}{n}.$$

This is just the k-th component of X plus ϵ/n. Hence,

$$F(X) \neq X,$$

and the lemma is proved.

THEOREM 1. *The Brouwer Fixed Point Theorem implies Sperner's Lemma.*

Proof. Let σ^n be an n-simplex, with a subdivision $\mathscr{S}$ and a proper labeling L. Construct the function f given by (5) and (6). This function is a continuous mapping of σ^n into itself; hence, by the Brouwer Fixed Point Theorem, f has a fixed point in σ^n.

By Lemma 1, any fixed point of f must be contained in an n-simplex $\sigma_i^n \in \mathscr{S}$ which has a complete set of labels on its vertices. Since X is a fixed point, such a σ_i^n must exist in $\mathscr{S}$, and Sperner's Lemma holds.

3. TUCKER'S LEMMA

As in the last section, we begin by stating the results whose equivalence will be proved:

FUNDAMENTAL NON-EXISTENCE THEOREM.[1] *There is no continuous mapping of an n-ball into an $(n-1)$-sphere sending each pair of antipodal points of the boundary of the ball into a pair of antipodal points on the sphere.*

To state Tucker's Lemma, following Tucker [9], we define $I(n, s)$ as the set of all points $X = (x_1, x_2, \ldots, x_n)$ in $\mathscr{R}^n$ which are *integer* solutions of the $2n$ inequalities

$$0 \leqslant x_1 \leqslant s, 0 \leqslant x_2 \leqslant s, \ldots, 0 \leqslant x_n \leqslant s,$$

where n and s are positive integers. The set $I(n, s)$ consists of all of the lattice points contained in a real n-cube of side s.

We will say that two points $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ contained in $I(n, s)$ are *adjoining* points if

$$| x_i - y_i | = 0 \text{ or } 1,$$

for all $i = 1, 2, \ldots, n$, and $X \neq Y$. If we view $I(n, s)$ as the vertices of a decomposition of the n-cube of side s into unit n-cubes, then two points are adjoining if they are both vertices of one of the unit n-cubes.

We will say that X and Y are *antipodal boundary points* if $x_i + y_i = s$, for all $i = 1, 2, \ldots, n$, and $x_i y_i = 0$ for some i. In terms of the n-cube of side s, this means that X and Y lie on its boundary, and both lie on some line passing through the point at the center of the cube.

Finally, we will define $\Delta_n = \{\pm 1, \pm 2, \ldots, \pm n\}$, and a *labeling* of $I(n, s)$ as a function

$$L\colon I(n, s) \rightarrow \Delta_n .$$

We will say that a labeling is *proper* if for every pair of antipodal boundary points X and X',

$$L(X) + L(X') = 0.$$

TUCKER'S LEMMA [8]. *There is no proper labeling of $I(n, s)$ such that for all adjoining points X and Y*

$$L(X) + L(Y) \neq 0.$$

In order to prove Tucker's Lemma from the fundamental non-existence theorem we assume the lemma is false, and we construct a mapping of the n-ball to the $(n-1)$-sphere which violates the non-existence theorem.

[1] This is a generalization to n-space of the *Fundamental Non-existence Theorem* as stated by Tucker [8]. This also appears in Lefschetz [5, p. 141, #7].

To do this, we will first map an n-ball onto an n-cube. Then, following Tucker [9], we will map the n-cube onto the boundary of the n-cross polytope. Finally we will map the boundary of the n-cross polytope onto an $(n-1)$-sphere. The n-cross polytope, $\mathrm{CP}(n)$ is the set of points $X \in \mathscr{R}^n$ satisfying the 2^n inequalities of the form

$$e_1 x_1 + e_2 x_2 + \cdots + e_n x_n \leqslant 1,$$

where $e_i = \pm 1$. Its extreme points are the $2n$ points $(\pm 1, 0, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, \pm 1)$.

We assume that we have found a proper labeling L of $I(n, s)$ such that for all adjoining X and Y, $L(X) + L(Y) \neq 0$. Let $C(n, s)$ be the n-cube of side s defined by

$$C(n, s) = \{X = (x_1, x_2, \ldots, x_n) \mid 0 \leqslant x_i \leqslant s, \text{ for } i = 1, 2, \ldots, n\}.$$

If $C(n, s)$ is subdivided into unit n-cubes, then $I(n, s)$ is the set of all of the vertices of all of n-cubes in the subdivision.

Let B_n be the n-ball of radius $s/2$ centered at the center of $C(n, s)$. We now define a function

$$f_1\colon B_n \to C(n, s)$$

in the following manner. Let $\|\cdot\|_{\max}$ denote the max norm in $\mathscr{R}^n$, and $\|\cdot\|$ denote the Euclidean norm. Then, for $X \in B_n$, we define

$$f_1(X) = \begin{cases} c + \dfrac{\| X - c \|}{\| X - c \|_{\max}} (X - c), & \text{for } X \neq c, \\ c, & \text{for } X = c, \end{cases}$$

where $c = (s/2, s/2, \ldots, s/2)$ is the center of B_n. This map is clearly continuous for all $X \neq c$ contained in B_n. To see that it is continuous at $X = c$, we observe that if $X = c + \epsilon u$, where u is a unit vector in $\mathscr{R}^n$. then

$$\lim_{\epsilon \to 0} f_1(X) = c.$$

We should observe that f_1 carries pairs of antipodal boundary points of B_n onto pairs of antipodal boundary points of $C(n, s)$. By a pair of antipodal boundary points of $C(n, s)$, we mean a pair of points X and Y which lie on the boundary of $C(n, s)$ and which both lie on some line passing through the center of the cube. If $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, they are *antipodal boundary points* if $x_i + y_i = s$ for all $i = 1, 2, \ldots, n$, and $x_i y_i = 0$ for some i. Notice that every pair of antipodal boundary points of $I(n, s)$ is also a pair of antipodal boundary points of $C(n, s)$.

For B_n, points X and Y are a pair of antipodal boundary points if they lie on the boundary and both lie on some diameter of B_n. That is,

$$\| X - c \| = \| Y - c \| = s/2, \text{ and } X - c = -Y + c \text{ (or } X + Y = 2c).$$

Then,

$$f_1(X) = c + \frac{\| X - c \|}{\| X - c \|_{\max}} (X - c)$$

$$= c - \frac{\| X - c \|}{\| Y - c \|_{\max}} (Y - c),$$

$$f_1(Y) = c + \frac{\| Y - c \|}{\| Y - c \|_{\max}} (Y - c).$$

Hence $f_1(X) + f_1(Y) = 2c = (s, s, \ldots, s)$, and, therefore,

$$(f_1(X))_i + (f_1(Y))_i = s, \qquad \text{for} \quad 1 \leqslant i \leqslant n. \tag{7}$$

Now, let j be an index such that

$$| y_j - s/2 | \geqslant | y_i - s/2 |, \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

It is easily seen that

$$(f_1(Y))_j (f_1(X))_j = s \cdot 0 = 0. \tag{8}$$

Thus, by (7) and (8), $f_1(X)$ and $f_1(Y)$ are antipodal boundary points of $C(n, s)$.

We will next construct a continuous function f_2 from $C(n, s)$ to the n-cross polytope, $CP(n)$. If we let $V^i = (v_1^i, \ldots, v_n^i)$ be defined by

$$v_j^i = \delta_{ij},$$

then the extreme points of $CP(n)$ are $\pm V^i$, $i = 1, 2, \ldots, n$. The $(n-1)$-boundary faces of $CP(n)$ are given as the convex hulls of all combinations of n of the extreme points $W^1, W^2, \ldots, W^n$, where

$$W^i = \text{either } V^i \text{ or } -V^i.$$

That is, every combination of n extreme points which contains no pair of antipodal extreme point V^i and $-V^i$ has an $(n-1)$-boundary face as its convex hull. There are 2^n such boundary faces.

We will construct f_2 by first defining it on $I(n, s)$ and then extending it to $C(n, s)$. Let $X \in I(n, s)$. Define

$$f_2(X) = \operatorname{sgn}(L(X))\, V^{|L(X)|}, \tag{9}$$

where $\operatorname{sgn}(r) =$ sign of r. This gives a mapping

$$f_2\colon I(n, s) \to \{\pm V^1, \pm V^2, \ldots, \pm V^n\}.$$

We now want to extend f_2 to $C(n, s)$ in such a way that for each point $X \in C(n, s)$, if X is contained in a unit n-cube of the subdivision having vertices $X_1, X_2, \ldots, X_{2^n} \in I(n, s)$, then $f_2(X) \in$ convex hull$\{f_2(X_1), f_2(X_2), \ldots, f_2(X_{2^n})\}$. One way to carry out this extension is to simplicially subdivide each unit n-cube into $n!$ n-simplexes as described by Lefschetz.[2]

The decomposition is carried out in the following manner: Let $C_1, C_2, \ldots, C_{s^n}$ be the unit n-cubes in the decomposition of $C(n, s)$. For each C_i, assume

$$C_i = \{X \mid k_i \leqslant x_i \leqslant k_{i+1},\, 0 \leqslant k_i \leqslant s - 1\}.$$

Let π be a permutation of $1, 2, \ldots, n$, and define

$$\Delta_{i,\pi} = \{X \mid 0 \leqslant (x_{\pi(1)} - k_{\pi(1)}) \leqslant (x_{\pi(2)} - k_{\pi(2)}) \leqslant \cdots \leqslant (x_{\pi(n)} - k_{\pi(n)}) \leqslant 1\}.$$

$\Delta_{i,\pi}$ is an n-simplex contained in C_i. If P_n is the permutation group on n letters, then

$$\{\Delta_{i,\pi}\}_{\pi \in P_n}$$

is a simplicial decomposition of C_i. The vertices of the simplex $\Delta_{i,\pi}$ are simply some $n + 1$ of the vertices of C_i. That is, they are some $n + 1$ elements of $I(n, s)$ which are all adjoining. If

$$\Delta = \{\Delta_{i,\pi}\}, \qquad 1 \leqslant i \leqslant s^n, \quad \pi \in P_n,$$

then Δ provides a decomposition of $C(n, s)$ into $n!\, s^n$ simplexes, where the extreme points of each element $\Delta_{i,\pi}$ are some $n + 1$ adjoining elements in $I(n, s)$.

This decomposition provides a unique representation of every point $X \in C(n, s)$ as a convex combination of $n + 1$ adjoining points in $I(n, s)$. That is, for $X \in C(n, s)$,

$$X = \sum_{j=1}^{n+1} \lambda_j X_j, \tag{10}$$

[2] See Lefschetz [5, p. 140, #3].

where $X_j \in I(n, s)$, for $j = 1, 2, \ldots, n+1$, $0 \leqslant \lambda_j \leqslant 1$, and

$$\sum_{j=1}^{n+1} \lambda_j = 1.$$

Here some of the λ_j may be zero if X lies on a boundary face common to several of the $\Delta_{i,\pi}$, but the representation in terms of the X_j having non-zero coefficients is unique. The representation (10) of X is simply gotten by writing X in the barycentric coordinates of the simplex $\Delta_{i,\pi}$ in which it lies.

We can now extend f_2 to all of $C(n, s)$. For this, let $X \in C(n, s)$ as given by (10). Define

$$f_2(X) = \sum_{j=1}^{n+1} \lambda_j f_2(X_j), \tag{11}$$

where $f_2(X_j)$ is defined by (9). The function

$$f_2\colon C(n, s) \to CP(n)$$

is clearly a continuous mapping.

We observe that $f_2(C(n, s)) \subset \partial(CP(n))$, where $\partial(CP(n))$ is the boundary of $CP(n)$. To see this, let $X \in C(n, s)$. Then, by (11),

$$f_2(X) = \sum_{j=1}^{n+1} \lambda_j f_2(X_j),$$

where $0 \leqslant \lambda_j \leqslant 1$, and $\sum_{j=1}^{n+1} \lambda_j = 1$, and the X_j, $1 \leqslant j \leqslant n+1$ are pairwise adjoining points of $I(n, s)$. By our assumption on L, for any pair X_i and X_j in this expansion,

$$L(X_i) + L(X_j) \neq 0,$$

and the set $\{f_2(X_1), f_2(X_2), \ldots, f_2(X_{n+1})\}$ contains no pair of antipodal boundary points. But

$$f_2(X) \in \operatorname{conv}\{f_2(X_1), f_2(X_2), \ldots, f_2(X_{n+1})\},$$

which is a boundary face of $CP(n)$ of dimension $n - 1$ or less. Hence,

$$f_2\colon C(n, s) \to \partial(CP(n)).$$

Finally, for f_2, we show that f_2 maps pairs of antipodal boundary points onto pairs of antipodal boundary points. For $CP(n)$, two points X and Y are *antipodal boundary points* if they both lie on the boundary, and lie on some line through the center of $CP(n)$ (the origin). That is, X and Y are antipodal boundary points of $CP(n)$ if

$$\sum_{i=1}^{n} | x_i | = \sum_{i=1}^{n} | y_i | = 1 \tag{12}$$

and

$$x_i + y_i = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n. \tag{13}$$

Let X and Y be antipodal boundary points of $C(n, s)$. Since $f_2(C(n, s)) \subset \partial(CP(n))$, $f_2(X)$ and $f_2(Y)$ are on the boundary of $CP(n)$, and (12) holds. Now, consider the representation of X given by (10); namely,

$$\sum_{j=1}^{n+1} \lambda_j X_j ,$$

where $X_j \in I(n, s)$ for $1 \leqslant j \leqslant n + 1$. Since $X_j \in I(n, s)$, it has coordinates $x_{j_1}, x_{j_2}, \ldots, x_{j_n}$ which are integer solutions of the $2n$ inequalities

$$0 \leqslant x_{j_i} \leqslant s, \ 1 \leqslant j \leqslant n.$$

Consider the points $Y_j = (y_{j_1}, \ldots, y_{j_n})$, $1 \leqslant j \leqslant n + 1$ defined by

$$y_{j_i} = s - x_{j_i} , \qquad \text{for} \quad 1 \leqslant j \leqslant n + 1, 1 \leqslant i \leqslant n.$$

Clearly, $Y_j \in I(n, s)$ for $1 \leqslant j \leqslant n + 1$. Also, since the X_j are adjoining, so are the Y_j. Now, since $x_i + y_i = s$, for $1 \leqslant i \leqslant n$,

$$Y = \sum_{j=1}^{n+1} \lambda_j Y_j .$$

Furthermore, since X is on the boundary of $C(n, s)$, it must have one component which is 0 or s, and each X_j must have this same value for that component. Therefore, each X_j is a boundary point. From the construction Y_j, it must be the antipodal boundary point to X_j, and therefore,

$$L(X_j) = -L(Y_j).$$

Hence

$$f_2(X_j) = -f_2(Y_j),$$

for $1 \leqslant j \leqslant n + 1$.

But

$$f_2(X) = \sum_{j=1}^{n+1} \lambda_j f_2(X_j) = -f_2(Y),$$

which implies that

$$(f_2(X))_i + (f_2(Y))_i = 0$$

for $1 \leqslant i \leqslant n$. Hence, $f_2(X)$ and $f_2(Y)$ are antipodal boundary points of $CP(n)$. To summarize, we have:

LEMMA 2. *The function*

$$f_2\colon C(n, s) \to CP(n)$$

defined by (9) *and* (11) *is a continuous mapping whose range is* $\partial(CP(n))$. *Further, it carries antipodal boundary points of* $C(n, s)$ *onto antipodal boundary points of* $CP(n)$.

To complete our construction, we will now map $\partial(CP(n))$ onto S_{n-1}, the $(n-1)$-sphere of radius 1. We define

$$f_3\colon \partial(CP(n)) \to S_{n-1}$$

by

$$f_3(X) = \frac{X}{\| X \|}.$$

This mapping is clearly continuous on $\partial(CP(n))$, and maps antipodal points of $\partial(CP(n))$ onto antipodal points of S_{n-1}. Finally, let

$$f = f_3 \circ f_2 \circ f_1,$$

$$f\colon B_n \to S_{n-1}.$$

By the construction of f_1, f_2, and f_3, the function f is continuous and carries antipodal boundary points onto antipodal boundary points. But, the existence of such a function f contradicts the fundamental non-existence theorem. Therefore, our assumption that a proper labeling L of $I(n, s)$ exists such that for all adjoining points X and Y

$$L(X) + L(Y) \neq 0$$

is contradicted, and Tucker's Lemma holds.

4. Another Labeling of $C(n, s)$

In this section we will describe a different labeling of $C(n, s)$, and derive a combinatorial result which is equivalent to Tucker's Lemma. We begin with a unit n-cube subdivision of $C(n, s)$. We will assume in this section that $s \geqslant 2$. Denote the unit n-cubes in the subdivision by C_j, $1 \leqslant j \leqslant s^n$, and let

$$\mathscr{C} = \{C_j\}_{1 \leqslant j \leqslant s^n}.$$

Let $V_n = \{(\pm 1, \pm 1, \ldots, \pm 1)\}$. This is a set of 2^n n-tuples, and can be viewed as the set of vertices of the n-cube consisting of all points $X = (x_1, x_2, \ldots, x_n)$ satisfying the $2n$ inequalities

$$-1 \leqslant x_i \leqslant 1,$$

for $1 \leqslant i \leqslant n$.

By an *n-labeling* of $C(n, s)$, we will mean a function

$$L\colon \mathscr{C} \to V_n.$$

We will say that a non-empty subset $S \subset \mathscr{C}$ is an *adjacency set* if

$$I(S) = \bigcap_{C \in S} C \neq \phi.$$

A set $S \subset \mathscr{C}$ is a *maximal adjacency set* if it is an adjacency set, and

$$I(S) \cap C = \phi,$$

for all $C \notin S$.

As in the last section, we will let $c = (s/2, s/2, \ldots, s/2)$ denote the center of $C(n, s)$. We will say that two unit n-cubes $C, C' \in \mathscr{C}$ are *antipodal boundary cubes* if

(i) $C \cap \partial(C(n, s)) \neq \phi$,

(ii) $C' \cap \partial(C(n, s)) \neq \phi$, and

(iii) $\{X - c \mid X \in C\} = \{-(Y - c) \mid Y \in C'\}$.

An n-labeling L of $C(n, s)$ will be called an adjacency labeling if for every maximal adjacency set S the following holds: Let

$$S = \{C_i\}_{1 \leqslant i \leqslant 2^n}$$

(note: every maximal adjacency set contains 2^n elements), and let

$$L(C_i) = (l_{i1}, l_{i2}, \ldots, l_{in}).$$

Then there is some index k such that

$$l_{i_k} = l_{j_k},$$

for all i, j, $1 \leqslant i, j \leqslant 2^n$.

We will say that an n-labeling L of $C(n, s)$ is *proper* if it satisfies the following condition:

If C and C' are antipodal boundary cubes, then $L(C) + L(C') = (0, 0,..., 0)$.

THEOREM 2. *There is no proper n-labeling of $C(n, s)$, $s > 2$, which is also an adjacency labeling.*

We will prove this theorem in a manner similar to the proof of Tucker's Lemma in the last section. The result will be derived from the following version for n-cubes of the fundamental non-existence theorem:

FUNDAMENTAL NON-EXISTENCE THEOREM (Cubical Form).[3] *There is no continuous mapping of an n-cube into the boundary of an n-cube sending each pair of antipodal points of the boundary of the n-cube into a pair of antipodal points of the range.*

We assume that there is a proper n-labeling L of $C(n, s)$, which is also an adjacency labeling. We will use this labeling to construct a mapping

$$f\colon C(n, s) \to C_n,$$

where C_n is the n-cube of side 2 centered at the origin. Notice V_n is the set of vertices of C_n. We will actually define f on a larger domain than $C(n, s)$. We first border $C(n, s)$ by unit n-cubes. That is, we take the set $BC(n, s)$, the bordered cube, to be

$$BC(n, s) = \{X \mid -1 \leqslant x_i \leqslant s + 1, 1 \leqslant i \leqslant n\}.$$

Let $\mathscr{B}$ be the set of unit n-cubes in the cubical subdivision of $BC(n, s)$. Note that $\mathscr{C} \subset \mathscr{B}$. For each element $C \in \mathscr{B} - \mathscr{C}$, there is a unique element $C' \in \mathscr{C}$ such that

$$\dim(C \cap C') > \dim(C \cap C''), \tag{14}$$

for all $C'' \in \mathscr{C}$, $C'' \neq C'$. The existence of this unique element C' is clear.

[3] This result is clearly equivalent to the version of the fundamental non-existence theorem for the n-ball stated in the last section.

We extend L to the elements of $\mathscr{B} - \mathscr{C}$ by letting $L(C) = L(C')$, where $C \in \mathscr{B} - \mathscr{C}$ and C' is the element of $\mathscr{C}$ satisfying (14).

Now, let

$$BC'(n, s) = \{X \mid -1/2 \leqslant x_i \leqslant s + 1/2, 1 \leqslant i \leqslant n\}.$$

We have

$$C(n, s) \subset BC'(n, s).$$

The function f will be defined on $BC'(n, s)$. For this, let $\mathscr{B}'$ be the set of elements of a unit n-cube subdivision of $BC'(n, s)$. The vertices of this subdivision are just the centers of all of the elements of $\mathscr{B}$. Let K be the set of centers of all elements $C \in \mathscr{B}$. We define f on K first. For $X \in K$, define

$$F(X) = L(C), \tag{15}$$

where X is the center of C. Using the simplicial subdivision described in the last section, subdivide $BC'(n, s)$ into $n!\,(s + 1)^n$ n-simplexes. Each simplex in the subdivision has as its extreme points $n + 1$ elements of K all of which are vertices of some element in $\mathscr{B}'$. Let $X \in BC'(n, s)$, then X has an expansion

$$X = \sum_{j=1}^{n+1} \lambda_j X_j\,, \tag{16}$$

where $0 \leqslant \lambda_j \leqslant 1$, $\sum_{j=1}^{n+1} \lambda_j = 1$, and $X_j \in K$. Define

$$f(X) = \sum_{j=1}^{n+1} \lambda_j f(X_j). \tag{17}$$

The function f given by (15) and (17) is clearly continuous.

We next observe that f maps $BC'(n, s)$ into the boundary of C_n. To see this, let $X \in BC'(n, s)$, then it has an expansion as in (16), where all of the elements X_j in the expansion are extreme points of some element C' in $\mathscr{B}'$. These elements are the centers of elements $C_1, C_2, \ldots, C_{n+1}$ in $\mathscr{B}$. The point at the center of C' lies in the intersection of $C_1, C_2, \ldots, C_{n+1}$, and, therefore, there is some maximal adjacency set S with

$$\{C_1, C_2, \ldots, C_{n+1}\} \subset S.$$

From the condition that L is an adjacency labeling, there is some entry common to $L(C_1), L(C_2), \ldots, L(C_{n+1})$. That is, there is some index i, $1 \leqslant i \leqslant n$, such that

$$(f(X_1))_i = (f(X_2))_i = \cdots = (f(X_{n+1}))_i = \alpha,$$

where $\alpha = \pm 1$. From (17)

$$f(X) = \sum_{j=1}^{n+1} \lambda_j f(X_j),$$

so

$$(f(X))_i = \left(\sum_{j=1}^{n+1} \lambda_j f(X_j) \right)_i = \alpha.$$

Hence, $(f(X))_i = \pm 1$, which implies that it lies on the boundary of C_n.

We finally show that f maps antipodal boundary points of $C(n, s)$ onto antipodal boundary points of C_n. For this, let X and Y be antipodal boundary points of $C(n, s)$. As was shown in the last section, if

$$X = \sum_{j=1}^{n+1} \lambda_j X_j, \tag{18}$$

then

$$Y = \sum_{j=1}^{n+1} \lambda_j (V - X_j), \tag{19}$$

where $V = (s, s, \ldots, s)$. Since $X \in \partial(C(n, s))$, the points X_j are the centers of elements $C_1, \ldots, C_{n+1} \in \mathscr{B}$, which are either

(i) elements of $\mathscr{C}$ which meet $\partial(C(n, s))$, or

(ii) elements of $\mathscr{B} - \mathscr{C}$.

In both cases, the point $V - X_j$ is the center of the element in $\mathscr{B}$ which is centrally symmetric to C_j. Thus, since L is a proper labeling, by applying (15), and the definition of the labeling on $\mathscr{B} - \mathscr{C}$, we have

$$f(V - X_j) = -f(X_j). \tag{20}$$

Applying (17) to (18) and (19), and employing (20),

$$f(X) = -f(Y). \tag{21}$$

Since

$$f\colon BC'(n, s) \to \partial(C_n),$$

$f(X)$ and $f(Y)$ are contained in $\partial(C_n)$, and, therefore, by (21), they are antipodal boundary points of C_n.

This completes the proof, since we have used L to construct a function f which contradicts the fundamental non-existence theorem, no such L can be found, and the theorem holds.

We should observe that it is not hard to demonstrate the equivalence of Theorem 2 and Tucker's Lemma. This equivalence is based on the fact that, if a proper n-labeling of $C(n, s)$ which is also an adjacency labeling exists, then a proper labeling of $I(n, s)$ such that

$$L(X) + L(Y) \neq 0$$

for all adjoining X, Y can be constructed from it, and vice versa. We omit the details of the constructions here.

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